

MULTIHOMOGENOUS NONNEGATIVE POLYNOMIALS AND SUMS OF SQUARES

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ABSTRACT. We refine and extend quantitative bounds, on the fraction of nonnegative polynomials that are sums of squares, to the multihomogenous case. In particular, we start laying the foundations for bounds incorporating arbitrary Newton polytopes. A key new ingredient is an isotropic measure introduced by zonal harmonics.

1. INTRODUCTION

Let $\mathbb{R}[\bar{x}] := \mathbb{R}[x_1, \dots, x_n]$ denote the ring of real n -variate polynomials and let $P_{n,2k}$ denote the vector space of forms (i.e. homogenous polynomials) of degree $2k$ in $\mathbb{R}[\bar{x}]$. A form $p \in P_{n,2k}$ is called *non-negative* if $p(\bar{x}) \geq 0$ for every $\bar{x} \in \mathbb{R}^n$. The set of *non-negative* forms in $P_{n,2k}$ is closed under nonnegative linear combinations and thus forms a cone. We denote cone of nonnegative forms of degree $2k$ by $\text{Pos}_{n,2k}$. A fundamental problem in polynomial optimization and real algebraic geometry is to efficiently certify non-negativity for real forms, i.e., membership in $\text{Pos}_{n,2k}$.

If a real form can be written as a sum of squares of other real polynomials then it is evidently non-negative. Polynomials in $P_{n,2k}$ that can be represented as sums of squares of real polynomials form a cone that we denote by $\text{Sq}_{n,2k}$. Clearly, $\text{Sq}_{n,2k} \subseteq \text{Pos}_{n,2k}$. We are then lead to the following natural question.

Question 1.1. *For which pairs of (n, k) do we have $\text{Sq}_{n,2k} = \text{Pos}_{n,2k}$?*

In his seminal 1888 paper [12] Hilbert showed that the answer to Question 1.1 is affirmative exactly for $(n, k) \in (\{2\} \times 2\mathbb{N}) \cup (\mathbb{N} \times \{2\}) \cup \{3, 4\}$. Hilbert's beautiful proof was not constructive: The first well known example of a non-negative form which is not sums of squares is due to Motzkin from around 1967: $x_3^6 + x_1^2 x_2^2 (x_1^2 + x_2^2 - 3x_3^2)$.

Hilbert stated a variation of Question 1.1 in his famous list of problems for 20th century mathematicians:

Hilbert's 17th Problem. *Do we have, for every n and k , that every $p \in \text{Pos}_{n,2k}$ is a sum of squares of rational functions?*

Artin and Schrier solved Hilbert's 17th Problem affirmatively around 1927 [1]. However there is no known efficient and general algorithm for finding the asserted collection of rational functions for a given input p . Despite the computational hardness of finding a representation as a sum of squares of rational functions, obtaining a representation as a sum of squares of *polynomials* (when possible) can be done efficiently via *semidefinite programming* (see, e.g., [15]). This connection to semidefinite programming (which has been used quite successfully in electrical engineering and optimization) strongly motivates a classification of which (n, k) have membership in $\text{Sq}_{n,2k}$ occuring with high probability, relative to some natural probability measure μ on $\text{Pos}_{n,2k}$.

1.1. From All or Nothing to Something in Between. Hilbert's 17th problem is essentially an algebraic problem. However methods from analysis have recently enabled

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some advances. The first example of this perspective is Gregoriy Blekherman's work: A consequence of his paper [6] is a probability measure μ on $\text{Pos}_{n,2k}$ supported in an hyperplane, for which $\mu(\text{Sq}_{n,2k}) \rightarrow 0$ as $n \rightarrow \infty$, for any fixed $k \geq 2$.

It is important to observe that for many problems of interest in algebraic geometry, forms with a special structure (e.g., sparse polynomials) behave differently from generic forms of degree $2k$. Precious little is known about Hilbert's 17th Problem in the setting of sparse polynomials [21, 17, 8]. So let us first recall the notion of Newton polytope and then a theorem of Reznick: For any $p(\bar{x}) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha$ with $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\bar{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, the *Newton polytope* of p is the convex hull $\text{Newt}(p) := \text{Conv}(\{\alpha \mid c_\alpha \neq 0\})$.

Theorem 1.1. [17, Thm. 1] *If $p = \sum_{i=1}^r g_i^2$ for some $g_1, \dots, g_r \in \mathbb{R}[\bar{x}]$ then $\text{Newt}(g_i) \subseteq \frac{1}{2}\text{Newt}(p)$ for all i .*

This theorem enables us to refine the comparison of cones of *sums of squares* and *non-negative* polynomials to be more sensitive to monomial term structure.

Definition 1.2. *For any polytope $Q \subset \mathbb{R}^n$ with vertices in \mathbb{Z}^n , let $N_Q := \#(Q \cap \mathbb{Z}^n)$, $c = (c_\alpha \mid \alpha \in Q \cap \mathbb{Z}^n)$, $p_c(\bar{x}) = \sum_{\alpha \in Q \cap \mathbb{Z}^n} c_\alpha \bar{x}^\alpha$ and then define*

$$\begin{aligned} \text{Pos}_Q &:= \{c \in \mathbb{R}^{N_Q} \mid p_c(\bar{x}) \geq 0 \text{ for every } x \in \mathbb{R}^n\} \\ \text{Sq}_Q &:= \{c \in \mathbb{R}^{N_Q} \mid p_c(\bar{x}) = \sum_i q_i(\bar{x})^2 \text{ where } \text{Newt}(q_i) \subseteq \tfrac{1}{2}Q\} \end{aligned} \quad \diamond$$

In our notation here, Blekherman's paper [6] focused on volumetric estimates for the cones $\text{Pos}_{Q_{n,2k}}$ and $\text{Sq}_{Q_{n,2k}}$, where $Q_{n,2k}$ is the scaled $(n-1)$ -simplex

$$\{\bar{x} \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 2k, x_1, \dots, x_n \geq 0\}.$$

In this context the following problem arises naturally:

Weighted Polytopal SOS Problem. *Given a polytope Q and a probability measure μ on Pos_Q , estimate $\mu(\text{Sq}_Q)$. \diamond*

Note that Hilbert's classic work [12] implies that, for any n and k and any continuous nonnegative probability measure on the cone $\text{Pos}_{n,2k}$, we have $\mu(\text{Sq}_{Q_{3,4}}) = \mu(\text{Sq}_{Q_{2,2k}}) = \mu(\text{Sq}_{Q_{n,2}}) = 1$. A related variant of the Weighted Polytopal SOS Problem was recently answered, as a consequence of the main theorem from [8]: There is now a complete combinatorial classification of those polytopes Q for which $\text{Sq}_Q = \text{Pos}_Q$.

1.2. Our Results. We focus on the *multihomogenous* case of the Weighted Polytopal SOS Problem. We have leaned toward general methods rather than ad hoc methods, in order to allow future study of arbitrary polytopes. We begin here with Cartesian products of scaled standard simplices.

Definition 1.3. *Assume henceforth that $n = n_1 + \dots + n_m$ and $k = k_1 + \dots + k_m$, with $k_i, n_i \in \mathbb{N}$ for all i , and set $N := (n_1, \dots, n_m)$ and $K := (k_1, \dots, k_m)$. We will partition the vector $\bar{x} = (x_1, \dots, x_n)$ into m sub-vectors $\bar{x}_1, \dots, \bar{x}_m$ so that \bar{x}_i consists of exactly n_i variables for all i , and say that $p \in \mathbb{R}[\bar{x}]$ is homogenous of type (N, K) if and only if p is homogenous of degree k_i with respect to \bar{x}_i for all i . Finally, let $Q_{N,K} := Q_{n_1,k_1} \times \dots \times Q_{n_m,k_m}$. \diamond*

Example 1.4. $p(\bar{x}) := x_1^3 x_4^2 + x_1 x_2^2 x_5^2 + x_3^3 x_4 x_5$ is homogenous of type (N, K) with $N = (3, 2)$ and $K = (2, 3)$. (So $\bar{x}_1 = (x_1, x_2, x_3)$ and $\bar{x}_2 = (x_4, x_5)$.) In particular, $\text{Newt}(p) \subseteq Q_{N,K} = Q_{3,2} \times Q_{2,3}$. \diamond

Multihomogenous forms appeared before in the work of Choi, Lam and Reznick. In particular they proved the following theorem in [7]:

Theorem 1.5. (Choi, Lam, Reznick) Let $N = (n_1, n_2, \dots, n_m)$ and $K = (2k_1, 2k_2, \dots, 2k_m)$ where $n_i \geq 2$ and $k_i \geq 1$ then $\text{Pos}_{Q_{N,K}} = \text{Sq}_{Q_{N,K}}$ if and only if $m = 2$ and (N, K) is either $(2, n_2; 2k_1, 2)$ or $(n_1, 2; 2, 2k_2)$.

Our result can be viewed as a localized version of Blekherman's Theorem and also as a quantitative version of the theorem of Choi, Lam and Reznick [7]. In order to state our result we need to introduce the following function on subsets of $P_{N,K}$.

Definition 1.6. Let S^{n-1} denote the standard unit $(n-1)$ -sphere in \mathbb{R}^n and define $S := S^{n_1-1} \times \dots \times S^{n_m-1}$, $P_{N,K} := \{p \in \mathbb{R}[\bar{x}] \text{ homogeneous of type } (N, K)\}$, and $C_{N,K} := \{p \in P_{N,K} \mid \int_S p \, d\sigma = 1\}$. For any $X \subseteq P_{N,K}$ we set $\mu(X) = \left(\frac{\text{vol}(X \cap C_{N,K})}{\text{vol}(B)} \right)^{\frac{1}{D_{N,K}}}$ where $D_{N,K}$ is the dimension of $P_{N,K}$ and B is the $D_{N,K}$ dimensional ball. \diamond

Our main theorem is the following.

Theorem 1.7. Let $N = (n_1, n_2, \dots, n_m)$ and $K = (2k_1, 2k_2, \dots, 2k_m)$. We define $L_{Q_{N,K}} := \{p \in \text{Pos}_{Q_{N,K}} : p = \sum_i l_{i1}^{2k_1} l_{i2}^{2k_2} \dots l_{im}^{2k_m} \text{ where } l_{ij} \text{ is a linear form in variables } \bar{x}_j\}$, then the following bounds hold

$$\begin{aligned} \frac{1}{4^m \sqrt{\max_i n_i}} \prod_{i=1}^m (2k_i + 1)^{-\frac{1}{2}} &\leq \mu(\text{Pos}_{Q_{N,K}}) \leq c_0 \\ c_1 \prod_{i=1}^m (2k_i + \frac{n_i}{2})^{-\frac{k_i}{2}} &\leq \mu(\text{Sq}_{Q_{N,K}}) \leq c_2 \prod_{i=1}^m (\frac{ck_i}{n_i + k_i})^{\frac{k_i}{2}} \\ c_3 \prod_{i=1}^m (2k_i + \frac{n_i}{2})^{-\frac{k_i}{2}} &\leq \mu(L_{Q_{N,K}}) \leq \sqrt{\max_i n_i} \prod_{i=1}^m 4(2k_i + 1)^{\frac{1}{2}} \prod_{i=1}^m (\frac{n_i}{2k_i})^{-\frac{k_i}{2}} \end{aligned}$$

where c_i are absolute constants with $c_0 \leq 5$ and $c = 2^{10}e$.

To compare our bounds with Blekherman's bounds [6] let us consider the following special cases of Theorem 1.7:

Corollary 1.8. (1) For $N = (2, n-2)$ and $K = (2k-2, 2)$ we have the following bounds:

$$c_1 (2k-1)^{\frac{-k+1}{2}} (n+2)^{\frac{-1}{2}} \leq \frac{\mu(\text{Sq}_{Q_{N,K}})}{\mu(\text{Pos}_{Q_{N,K}})} \leq 1$$

(2) Assume $n = k \cdot n_1$, we partition into k groups by setting $N = (n_1, n_1, \dots, n_1)$ and $K = (2, 2, \dots, 2)$, then we have the following bounds:

$$c_1 (2 + \frac{n}{2k})^{\frac{-k}{2}} \leq \frac{\mu(\text{Sq}_{Q_{N,K}})}{\mu(\text{Pos}_{Q_{N,K}})} \leq c_2 (\frac{n}{ck})^{\frac{-k+1}{2}}$$

where c , c_1 and c_2 are absolute constants with $c = \frac{2^{10}e}{48}$.

Note that both cases considered above are contained in $Q_{n,2k}$. In particular, Blekherman's Theorems 4.1 and 6.1 from [6] give the following estimates:

$$\frac{n^{\frac{k+1}{2}}}{(\frac{n}{2} + 2k)^k} \frac{c_1 k! (k-1)!}{4^{2k} (2k)!} \leq \frac{\mu(\text{Sq}_{Q_{n,2k}})}{\mu(\text{Pos}_{Q_{n,2k}})} \leq \frac{c_2 4^{2k} (2k)! \sqrt{k}}{k!} n^{-\frac{k+1}{2}}$$

where c_1 and c_2 are absolute constants.

The first case in Corollary 1.8, is a modest example to show the reflection of monomial structure in our bounds. Bounds in the second case is dependent on $\frac{n}{k}$ instead of n which shows the effect of underlying multihomogeneity. In particular in the cases that k and n are comparable bounds behave significantly different then the bounds of Blekherman.

In general, Theorem 1.7 proves that if we assume multihomogeneity on the set of variables $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m$, bounds derived in Blekherman's work for the ratio of sums of squares to non-negative polynomials, holds locally for every set of variable \bar{x}_j .

The remainder of this paper is structured as follows: In Section 2 we define two different inner products, and investigate basic relations between geometries introduced by these two inner products. Section 2 also includes definition of zonal harmonics and their basic properties. The hurried reader can see the definitions at the very beginning and then go to Lemmata 2.8, 2.11, 2.13 and 2.14. In Section 3 we prove volumetric bounds for $\text{Pos}_{Q_{N,K}}$. A key step is discovering existence of an isotropic measure linked to the zonal harmonics. In Section 4 we give bounds for $\text{Sq}_{Q_{N,2K}}$ via classical convex geometry. Section 5 is devoted to polynomials that are powers of linear forms. The bounds there are derived by a simple duality observation.

2. HARMONIC POLYNOMIALS AND EUCLIDEAN BALLS

In this section we develop necessary background for the proof of Theorem 1.7. We are going to make use of two different inner products on $P_{N,K}$, mainly due to two useful notions of duality.

Definition 2.1. For $f, g \in P_{N,K}$ the “usual” inner product is defined as $\langle f, g \rangle := \int_S fg \, d\sigma$. For $f(x) = \sum_{\alpha} c_{\alpha} x^{\alpha} \in P_{N,K}$ with $\alpha = (\alpha_1, \dots, \alpha_n)$ we define the linear differential operator $D[f] := \sum_{\alpha} c_{\alpha} \left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \right)$, and set $\langle f, g \rangle_D := D[f](g)$. This way of defining $\langle f, g \rangle_D$, introduces an inner product which we call the “differential” inner product. \diamond

If $g(x) = \sum_{\alpha} b_{\alpha} x^{\alpha} \in P_{n,d}$ and $f(x) = \sum_{\alpha} c_{\alpha} x^{\alpha} \in P_{n,d}$ then it is easily checked that

$$\langle f, g \rangle_D = d! \sum_{\alpha_1 + \dots + \alpha_n = d} \frac{c_{\alpha} b_{\alpha}}{\binom{d}{\alpha_1, \dots, \alpha_n}}$$

where $\binom{d}{\alpha_1, \dots, \alpha_n}$ is the multinomial coefficient $\frac{d!}{\alpha_1! \cdots \alpha_n!}$.

Below we list some useful properties of the differential inner product. This inner product is actually the *Bombieri-Weyl inner product* (see, e.g., [19]) in our setting.

Lemma 2.2. For any vectors $\bar{v}, \bar{x} \in S$ we define

$$K_v(x) = (v_1 x_1 + \dots + v_{n_1} x_{n_1})^{2k_1} (v_{n_1+1} x_{n_1+1} + \dots + v_n x_{n_1+n_2})^{2k_2} \cdots (v_{n-n_m+1} x_{n-n_m+1} + \dots + v_n x_n)^{2k_m}.$$

Then

$$(1) \quad \langle p, K_v(x) \rangle_D = p(v)$$

- (2) $\langle pq, h \rangle_D = \frac{\deg(q)!}{\deg(h)!} \langle q, D[p](h) \rangle_D$
- (3) Let $s_i := \sum_{j=1}^i n_j$ and $s_0 := 0$ let for any i $\Delta_i = \frac{\partial^2}{\partial x_{s_{i-1}+1}^2} + \dots + \frac{\partial^2}{\partial x_{s_i}^2}$ and $r_i = (x_{s_{i-1}+1}^2 + \dots + x_{s_i}^2)^{\frac{1}{2}}$ then $(p+2)(p+1)\langle r_i^2 p, q \rangle_D = \langle p, \Delta_i(q) \rangle_D$

We would like to compare the geometry induced by the two different inner products. For instance how are the Euclidean balls with respect to two different products related to each other? To find out, we will need to introduce a generalization of harmonic polynomials that applies to our multihomogenous setting, and define corresponding linear operators acting on underlying vector spaces.

We call f Π -harmonic if $\Delta_1(f) = \Delta_2(f) = \dots = \Delta_m(f) = 0$. One may suspect that Π -harmonicity is too strong a condition to be as natural as ordinary harmonicity. However, one observes that Π -harmonicity is a special case of Helgason's general theory of Harmonic polynomials (see Chapter 3, [13]).

Let $H_{N,K}$ be the space of Π -harmonic polynomials in $P_{N,K}$. We observe that $H_{N,K}$ is a vector space by the linearity of the Δ_i operators. Also, for any vector $K = (k_1, \dots, k_m) \in (\mathbb{N} \cup \{0\})^m$ we define

$$\mathcal{K}_K = \{Q \in (\mathbb{N} \cup \{0\})^m \mid Q = (q_1, \dots, q_m), 2k_i \geq q_i \text{ and } q_i \text{ is even.}\}.$$

Lemma 2.3. *For $U, Q \in \mathcal{K}_K$ such that $Q \neq U$ we have that $H_{N,U}$ and $H_{N,Q}$ are orthogonal with respect to the usual inner product.*

Proof. Let $f \in H_{N,Q}$ and $g \in H_{N,U}$ with $Q \neq U$. Without loss of generality, assume $Q = (q_1, \dots, q_m)$, $U = (u_1, \dots, u_m)$ and $q_1 \neq u_1$. Then we have

$$\begin{aligned} (q_1 - u_1) \int_{S^{n_1-1}} fg \, d\sigma_1 &= \int_{S^{n_1-1}} (q_1 fg - u_1 fg) \, d\sigma_1 \\ \int_{S^{n_1-1}} (q_1 fg - u_1 fg) \, d\sigma_1 &= \int_{S^{n_1-1}} (f D_n(g) - g D_n(f)) \, d\sigma_1 = \int_{B^{n_1-1}} (f \Delta_1 g - g \Delta_1 f) \, d\sigma_1 = 0. \end{aligned}$$

So $\int_{S^{n_1-1}} fg \, d\sigma_1 = 0$ and thus

$$\int_S fg \, d\sigma = \int_{S^{n_m-1}} \dots \int_{S^{n_1-1}} fg \, d\sigma_1 \dots d\sigma_m = 0$$

■

To see the connection between our two inner products we use a variant of a map introduced by Reznick [18]: Let $T : P_{N,K} \rightarrow P_{N,K}$ be defined via

$$T(f)(x) = A^{-1} \int_{S^{n_1-1} \times \dots \times S^{n_m-1}} f(v) K(v, x) \, d\sigma(v)$$

where $A = \int_{S^{n_1-1} \times S^{n_2-1} \times \dots \times S^{n_m-1}} x_{n_1}^{2k_1} x_{n_1+n_2}^{2k_2} \dots x_n^{2k_m}$ and $K(v, x)$ as defined in Lemma 2.2.

The following lemma shows that the operator T captures the relationship between our two inner products.

Lemma 2.4. *For all $f, g \in P_{N,K}$ we have $\langle T(f), g \rangle_D = A^{-1} 2k_1! \cdots 2k_m! \langle f, g \rangle$.*

Proof. The case of arbitrary m is only notationally more difficult than the $m=2$, thanks to induction. So we assume without loss of generality than $m=2$.

$$\langle T(f), g \rangle_D = \left\langle A^{-1} \int_{S^{n_1-1} \times S^{n_2-1}} f(v) v^K d\sigma, g \right\rangle_D = A^{-1} \int_{S^{n_1-1} \times S^{n_2-1}} \langle f(v) v^K, g \rangle_D d\sigma$$

Since $\langle v^K, g \rangle_D = 2k_1! 2k_2! g(v)$ we have

$$A^{-1} \int_{S^{n_1-1} \times S^{n_2-1}} \langle f(v) v^K, g \rangle_D d\sigma = A^{-1} \int_{S^{n_1-1} \times S^{n_2-1}} 2k_1! 2k_2! f(v) g(v) d\sigma = \frac{2k_1! 2k_2!}{A} \langle f, g \rangle$$

■

The lemma above immediately tells us T is one-to-one since for $f, h \in P_{N,K}$, assuming $T(f) = T(h)$ implies $\langle f, g \rangle = \langle h, g \rangle$ for all $g \in P_{N,K}$. To prove our next lemma we need to recall a theorem of Funk and Hecke.

Theorem 2.5. [20] *Given any measurable function K on $[-1, 1]$ such that the integral*

$$\int_{-1}^1 |K(t)| (1-t^2)^{\frac{n-2}{2}} dt$$

is well-defined, for every function H that is harmonic on S^n we have

$$\int_{S^n} K(\sigma \cdot \zeta) H(\zeta) d\zeta = \left(\text{Vol}(S^{n-1}) \int_{-1}^1 K(t) P_{k,n}(t) (1-t^2)^{\frac{n-2}{2}} dt \right) H(\sigma)$$

where $P_{k,n}(t)$ is the classical Gegenbauer (ultraspherical) polynomial.

Gegenbauer polynomials are naturally introduced by zonal harmonics and they do exist more generally in spaces of sparse polynomials as well. Zonal harmonics and ultraspherical polynomials in our setting are introduced in Lemmata 2.12, 2.13, 2.14.

Lemma 2.6. *For $f \in H_{N,K}$ we have*

$$T(f)(x) = E_{N,K} f(x)$$

where $E_{N,K}$ depends only on N and K .

Proof. Defining $K_i(v, x) = (v_{s_{i-1}+1} x_{s_{i-1}+1} + \cdots + v_{s_{i-1}+n_i} x_{s_{i-1}+n_i})^{2k_i}$, we have

$$T(f)(x) = A^{-1} \int_{S^{n_m-1}} \cdots \int_{S^{n_1-1}} f(v) \prod_{i=1}^m K_i(v, x) d\sigma_i(v)$$

We observe that each K_i satisfies the assumptions of the Funk-Hecke formula. So we have

$$T(f)(x) = A^{-1} \int_{S^{n_m-1}} \cdots \int_{S^{n_2-1}} A_1 f_1(v, x) \prod_{i=2}^m K_i(v, x) d\sigma_i(v)$$

here $f_1(v, x)$ means first n_1 variables are fixed to x_1, \dots, x_{n_1} and the rest left as variables in the integral and $A_1 = \left(\frac{\text{vol}(S^{n_1-1})}{\dim(H_{n_1, 2k_1})} \int_{-1}^1 t^{2k_1} P_{n_1, 2k_1}(t) (1-t^2)^{\frac{n_1-2}{2}} dt \right)$ where $P_{n_1, 2k_1}$ is the corresponding ultraspherical polynomial. Iterating the argument we have

$$T(f)(x) = A^{-1} \prod_i A_i f(x) = A^{-1} \prod_i \left(\frac{\text{vol}(S^{n_i-1})}{\dim(H_{n_i, 2k_i})} \int_{-1}^1 t^{2k_i} P_{n_i, 2k_i}(t) (1-t^2)^{\frac{n_i-2}{2}} dt \right) f(x)$$

■

Thanks to Lemma 2.6 we know that Π -harmonic polynomials are eigenvectors for T . We also know a relation between our two inner products thanks to Lemma 2.4, and the orthogonality of spaces of Π -harmonic polynomials with respect to usual inner product thanks to Lemma 2.3. We thus immediately obtain the following corollary.

Corollary 2.7. *For $U, Q \in \mathcal{K}_K$ such that $Q \neq U$ we have that $H_{N,U}$ and $H_{N,Q}$ are orthogonal with respect to the differential inner product.*

Remark 2.1. *For notational convenience we set $r = r_1 \cdots r_m$ where r_i is as defined in Lemma 2.2 and let $r^\alpha = r_1^{\alpha_1} \cdots r_m^{\alpha_m}$ for any $\alpha \in \mathbb{Z}^m$.*

For $g \in SO(n_1) \times \cdots \times SO(n_m)$ and $f \in P_{N,K}$ we define $g \circ f$ by setting $g \circ f(x) = f(g^{-1}(x))$. This gives a well defined group action on vector spaces of polynomials. We observe that the operator T commutes with $SO(n_1) \times \cdots \times SO(n_m)$ action.

This implies that $T(r^K) = ar^K$ for some constant a since the only polynomials that are fixed under the action are constant multiples of r^K . To compute a we check $1 = T(r^K)(e_{n_1} + \cdots + e_n) = ar^K(e_1 + \cdots + e_n) = a$. Hence r is fixed under T . Lemma 2.4 implies that the hyperplane orthogonal to r is fixed also. \diamond

Now we prove a decomposition lemma for $P_{N,K}$ in a Hilbert space sense.

Lemma 2.8. $P_{N,K} = \bigoplus_{\alpha \in \mathcal{K}_K} r^{K-\alpha} H_{N,\alpha}$

Proof. First observe that $r^{K-\alpha} H_{N,\alpha} \perp r^{K-\beta} H_{N,\beta}$ for any $\alpha \neq \beta$ by Lemma 2.3. Also via iterated usage of third property in Lemma 2.2 we observe $r^{K-\alpha} H_{N,\alpha} \perp_D r^{K-\beta} H_{N,\beta}$. So direct sum makes sense for both of the inner products. Let $E = \bigoplus_{\alpha \in \mathcal{K}_K} r^{K-\alpha} H_{N,\alpha}$ and assume $E \neq P_{N,K}$. This implies there exists $p \in P_{N,K}$ such that $p \perp_D E$. By assumption $p \notin H_{N,K}$ therefore there exist i such that $\Delta_i(p) \neq 0$. Wlog say $\Delta_1(p) = p_1 \neq 0$. Then if p_1 is Π -harmonic we have $\langle p, r_1^2 p_1 \rangle_D = \langle \Delta_1(p), p_1 \rangle_D = \langle p_1, p_1 \rangle_D \neq 0$. This yields a contradiction since $r_1^2 p_1 \in E$. If p_1 is not Π -harmonic there exist a j such that $\Delta_j(p_1) \neq 0$. Wlog say $\Delta_1(p_1) = p_2$. Repeating the same argument, we arrive to a contradiction surely since all polynomials of degree 0 are Π -harmonic! ■

Corollary 2.9. $P_{N,K}(S) = \bigoplus_{\alpha \in K^m} H_{N,\alpha}(S)$ where $P_{N,K}(S)$ is restriction of $P_{N,K}$ to $S = S^{n_1-1} \times S^{n_2-1} \times S^{n_3-1} \times \cdots \times S^{n_m-1}$.

From Lemma 2.8 we know how $P_{N,K}$ is decomposed into Π -harmonic polynomials. We also know $T(f) = E_{N,K} f$ for any $f \in H_{N,K}$. For $f \in r^{K-\alpha} H_{N,\alpha}$, since T is averaging over S where $r^{K-\alpha}$ is constant, repeating Funk-Hecke argument in Lemma 2.6 we observe $T(f) = C_{N,\alpha} f$ for some constant depending on N and α only. We would like to compute these constants and write the operator T explicitly. Thankfully the integrals that gives the constants are

well known and computed. (See for example Lemma 7.4 of [5]). We write the result without proof.

Lemma 2.10. *Let f_α be the projection of f onto the subspace $H_{N,\alpha}$ then we have*

$$T(f) = \sum_{\alpha \in \mathcal{K}_K} \prod_{i=1}^m \frac{k_i! \Gamma(\frac{n_i+2k_i}{2})}{(k_i - \frac{\alpha_i}{2})! \Gamma(\frac{n_i+2k_i+\alpha_i}{2})} r^{K-\alpha} f_\alpha.$$

Now let $f = \sum_{\alpha \in \mathcal{K}_K} r^{K-\alpha} f_\alpha$ that is $\|f\|^2 = \langle f, f \rangle = \sum_{\alpha \in \mathcal{K}_K} \|f_\alpha\|^2$. We set $a_\alpha = \prod_{i=1}^m \frac{k_i! \Gamma(\frac{n_i+2k_i}{2})}{(k_i - \frac{\alpha_i}{2})! \Gamma(\frac{n_i+2k_i+\alpha_i}{2})}$. Then define $A_{N,K} = \max_{\alpha \in \mathcal{K}_K} a_\alpha$, $B_{N,K} = \min_{\alpha \in \mathcal{K}_K} a_\alpha$ and $C_{N,K} = A^{-1} \prod_{i=1}^m 2k_i!$.

$$A_{N,K} C_{N,K} \|f\|^2 \geq \langle T(f), T(f) \rangle_D = C_{N,K} \langle f, T(f) \rangle \geq B_{N,K} C_{N,K} \|f\|^2$$

We denote the ball with respect to the usual inner product by B and the ball with respect to the differential inner product by B_D . The observation above implies

$$\frac{1}{\sqrt{A_{N,K} C_{N,K}}} T(B) \subseteq B_D \subseteq \frac{1}{\sqrt{B_{N,K} C_{N,K}}} T(B)$$

Lemma 2.11. *Let T be the operator on $P_{N,K}$ as defined before Lemma 2.4, let $A_{N,K}$, $B_{N,K}$, $C_{N,K}$ be defined as above, and let \dim be the dimension of $P_{N,K}$, then we have the following*

$$\begin{aligned} A_{N,K} &= 1 \\ B_{N,K} &= \left(\prod_{i=1}^m \binom{\frac{n_i}{2} + 2k_i}{k_i} \right)^{-1} \\ \prod_{i=1}^m \left(\frac{1}{2k_i + \frac{n_i}{2}} \right)^{\frac{k_i}{2}} &\leq |\det(T)|^{\frac{1}{\dim}} \leq \prod_{i=1}^m \left(\frac{1}{1 + \frac{n_i}{2k_i}} \right)^{\frac{k_i}{2}} \\ \prod_{i=1}^m \left(\frac{1}{2k_i + \frac{n_i}{2}} \right)^{\frac{k_i}{2}} &\leq \sqrt{C_{N,K}} \left(\frac{|B_D|}{|B|} \right)^{\frac{1}{\dim}} \leq e^{\frac{k}{2}} \prod_{i=1}^m \left(1 + \frac{1}{\frac{n_i}{2k_i} + 1} \right)^{\frac{k_i}{2}} \end{aligned}$$

Proof.

$$a_\alpha = \prod_{i=1}^m \frac{k_i! (\frac{n_i}{2} + k_i)!}{(k_i - \frac{\alpha_i}{2})! (\frac{n_i}{2} + k_i + \frac{\alpha_i}{2})!}$$

It is quite clear that a_α is maximized for $\alpha = (0, 0, \dots, 0)$ and minimized for $\alpha = K$. Thus $A_{N,K} = 1$ and $B_{N,K} = \prod_{i=1}^m \left(\binom{\frac{n_i}{2} + 2k_i}{k_i} \right)^{-1}$. Also T is a diagonal operator in the basis of harmonic polynomials and its entries are a_α . Thus

$$\frac{|T(B)|}{|B|} = |\det(T)| = \left| \prod_{\alpha \in \mathcal{K}_K} a_\alpha^{\dim(H_{n,\alpha})} \right|$$

We observe $|\mathcal{K}_K| = (k_1 + 1)(k_2 + 1) \cdots (k_m + 1)$,

$$a_\alpha = \prod_{i=1}^m \frac{\binom{\frac{n_i}{2} + 2k_i}{k_i - \frac{\alpha_i}{2}}}{\binom{\frac{n_i}{2} + 2k_i}{k_i}}$$

this yields the formula

$$|\det(T)|^{\frac{1}{\dim}} = B_{N,K} \prod_{\alpha \in \mathcal{K}_K} \left(\prod_{i=1}^m \binom{\frac{n_i}{2} + 2k_i}{k_i - \frac{\alpha_i}{2}} \right)^{\frac{\dim(H_{N,\alpha})}{\dim(P_{N,K})}}$$

If we partition \mathcal{K}_K into $k_1 + 1$ subsets by defining $\mathcal{K}_j := \{\alpha \in \mathcal{K}_K : \alpha_1 = 2k_1 - 2j\}$ then we have

$$|\det(T)|^{\frac{1}{\dim}} = B_{N,K} \prod_{j=0}^{k_1} \prod_{\alpha \in \mathcal{K}_j} a_\alpha = B_{N,K} \prod_{j=0}^{k_1} \binom{\frac{n_1}{2} + 2k_1}{j}^{\frac{1}{k_1+1}} A$$

For some A determined by n_i and k_i for $i \geq 2$. We repeat the same trick for A and do some housekeeping to arrive at the following formula

$$\begin{aligned} |\det(T)|^{\frac{1}{\dim}} &= B_{N,K} \prod_{i=1}^m \left(\prod_{j=0}^{k_i} \binom{\frac{n_i}{2} + 2k_i}{j} \right)^{\frac{1}{k_i+1}} = \prod_{i=1}^m \left(\prod_{j=0}^{k_i} \frac{\binom{\frac{n_i}{2} + 2k_i}{j}}{\binom{\frac{n_i}{2} + 2k_i}{k_i}} \right)^{\frac{1}{k_i+1}} \\ |\det(T)|^{\frac{1}{\dim}} &= \prod_{i=1}^m \left(\prod_{j=0}^{k_i} \frac{k_1! (\frac{n_1}{2} + k_1)!}{j! (\frac{n_1}{2} + 2k_1 - j)!} \right)^{\frac{1}{k_i+1}} = \prod_{i=1}^m \prod_{j=1}^{k_i} \left(\frac{j}{\frac{n_1}{2} + 2k_1 - j + 1} \right)^{\frac{j}{k_i+1}} \end{aligned}$$

Applying the trivial bounds $(\frac{1}{\frac{n_1}{2} + 2k_1})^j \leq \frac{k_1! (\frac{n_1}{2} + k_1)!}{j! (\frac{n_1}{2} + 2k_1 - j)!}$ and $\frac{j}{\frac{n_1}{2} + 2k_1 - j + 1} \leq \frac{k_i}{\frac{n_i}{2} + k_i + 1}$ we have

$$\prod_{i=1}^m \left(\frac{1}{\frac{n_i}{2} + 2k_i} \right)^{\frac{k_i}{2}} \leq |\det(T)|^{\frac{1}{\dim}} \leq \prod_{i=1}^m \left(\frac{k_i}{\frac{n_i}{2} + k_i + 1} \right)^{\frac{k_i}{2}}$$

Let us note that

$$B_{N,K}^{-1/2} \leq \prod_{i=1}^m \left(e \cdot \frac{\frac{n_i}{2} + 2k_i}{k_i} \right)^{\frac{k_i}{2}}$$

$$B_{N,K}^{-1/2} |\det(T)|^{\frac{1}{\dim}} \leq e^{\frac{k}{2}} \prod_{i=1}^m \left(1 + \frac{1}{\frac{n_i}{2k_i} + 1} \right)^{\frac{k_i}{2}}$$

■

We define a class of Π -harmonic polynomials which turns out to be very useful. Assume y_1, \dots, y_M is an orthonormal basis of $H_{N,K}$. Then let $v \in S^{n_1-1} \times \dots \times S^{n_m-1}$, $q_v = \sum_{i=1}^M y_i(v) y_i$. We observe $q_v \in H_{N,K}$ moreover $\langle f, q_v \rangle = f(v)$ for all $f \in H_{N,K}$. This special polynomial q_v is called the zonal harmonic corresponding to the vector v on $H_{N,K}$. The following lemma states basic properties of zonal harmonics:

Lemma 2.12.

- (1) $q_v(w) = q_w(v)$ for all $v, w \in S$.
- (2) $q_w(v) = q_{T(w)}(T(v))$ for all $v, w \in S$ and $T \in O(n_1) \times O(n_2) \times O(n_3) \times \cdots \times O(n_m)$.
- (3) $q_v(v) = \|q_v\|^2 = \dim H_{N,K}$.
- (4) $|q_v(w)| \leq \dim H_{N,K}$.

Proof. (1) Let e_1, \dots, e_l be an orthonormal basis for $H_{N,K}$. Then

$$q_v = \sum_{i=1}^l \langle e_i, q_v \rangle e_i = \sum_{i=1}^m e_i(v) e_i. \text{ So } q_v(w) = \sum e_i(v) e_i(w) = q_w(v).$$

(2) For $p \in H_{N,K}$ we have

$$p(T(w)) = \langle p \circ T, q_w \rangle = \int_S p(T(x)) q_w(x) d\sigma(x) = \int_S p(x) q_w(T^{-1}(x)) d\sigma(x)$$

Since the zonal harmonic is unique we deduce that $q_w \circ T^{-1} = q_{T(w)}$.

(3) By the notation of (1) and using (2) afterward

$$\begin{aligned} q_v(v) &= \langle q_v, q_v \rangle_\Pi = \langle \sum e_i(v) e_i, \sum e_i(v) e_i \rangle_\Pi = \sum |e_i(v)|^2 \\ q_v(v) &= \int_S q_v(v) d\sigma(v) = \int_S \sum_{i=1}^m |e_i(v)|^2 d\sigma(v) = m = \dim H_{N,K} \end{aligned}$$

(4) $|\langle q_v, q_w \rangle| = |q_v(w)| \leq \|q_v\| \|q_w\| = \dim H_{N,K}$. Thus $|q_v(w)| \leq \dim H_{N,K} = q_v(v)$. ■

Now we define, for any vector $v \in S$, the polynomial $p_v = \sum_{\alpha \in \mathcal{K}_K} r^{K-\alpha} q_{v,\alpha}$ where $q_{v,\alpha}$ is the zonal harmonic corresponding to v in $H_{N,\alpha}$. Let $\|f\|_\infty = \max_{v \in S} |f(v)|$. We observe p_v inherits properties from zonal harmonics:

- Lemma 2.13.** (1) $f(v) = \langle f, p_v \rangle$ for every $f \in P_{N,K}$.
 (2) $\|p_v\|^2 = \langle p_v, p_v \rangle = \sum \langle q_{v,\alpha}, q_{v,\alpha} \rangle = \sum \dim H_{N,\alpha} = \dim P_{N,K}$
 (3) $|f(v)| = |\langle f, p_v \rangle| \leq \|f\| \|p_v\|$ and thus $\frac{\|f\|_\infty}{\|f\|} \leq \|p_v\| = \sqrt{\dim P_{N,K}}$.
 (4) $|p_v(w)| \leq p_v(v)$ for all $w \in S$ and $\frac{\|p_v\|_\infty}{\|p_v\|} = \sqrt{\dim P_{N,K}}$.

The third property turns out to be a characterization of the polynomials p_v :

Lemma 2.14. Let $f \in P_{N,K}$ be such that $\frac{\|f\|_\infty}{\|f\|} \geq \frac{\|g\|_\infty}{\|g\|}$ for all $g \in P_{N,K}$. Then f is a constant multiple of p_v for some v .

Proof. Assume $\|f\|_\infty = f(v) = C$ and let $T = \{g \in P_{N,K} : g(v) = C\}$. We observe for $g \in T$, $\|g\|_\infty \geq C$.

Using the assumption on f we deduce $\|g\| \geq \|f\|$ for all $g \in T$. Thus f is the shortest form on the hyperplane. We also observe $g(v) = \langle g, p_v \rangle$ from Lemma 2.13. This proves f to be a constant multiple of p_v . ■

Let us consider the hyperplane $L_C := \{q \in P_{N,K} : q(v) = C\} = \{q \in P_{N,K} : \langle q, p_v \rangle = C\}$. We define $SO(v) := \{g \in SO(n_1) \times \cdots \times SO(n_m) : g(v) = v\}$. Now observe that L_C is fixed under $SO(v)$ action. This implies p_v is fixed under $SO(v)$ action and thus, for every $c \in \mathbb{R}$ and $M_c = \{x \in \mathbb{R}^n : \langle x, v \rangle = c\}$, p_v is constant on M_c . This implies $p_v(w) = q_{N,K}(\langle v, w \rangle)$ for some univariate polynomial $q_{N,K}$. This $q_{N,K}$ is the Gegenbauer or ultraspherical polynomial

in our setting. Gegenbauer polynomial in our setting or the classical Gegenbauer polynomial will be both referred as ultraspherical polynomial throughout this paper.

3. THE CONE OF NONNEGATIVE POLYNOMIALS

In this section we construct an isotropic measure introduced by the zonal harmonics, then use a theorem of Lutwak, Yhang and Zhang (Theorem 3.1 below) on the volume of convex hull of an isotropic measure supported on the sphere. Our upper bound for $\text{Vol}(\overline{\text{Pos}}_{N,K})$ then follows from Theorem 3.1 via duality.

Let us start by defining isotropicity: A measure Z on S^{t-1} is isotropic if for every $x \in \mathbb{R}^t$ we have

$$\|x\|_2^2 = \int_{S^{t-1}} \langle x, y \rangle^2 dZ(y)$$

We need to introduce one more definition to state the theorem Lutwak, Yhang and Zhang. For a convex body $K \in \mathbb{R}^n$ the polar of K denoted by K° is defined as follows:

$$K^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}$$

The main theorem of [14] is the following:

Theorem 3.1. (*Lutwak, Yhang, Zhang*) [14] *If Z is an isotropic measure on S^{t-1} whose centroid is at the origin and $Z_\infty = \text{Conv}(\text{Supp}(Z))$, then we have*

$$\text{Vol}(Z_\infty^\circ) \leq \frac{t^{\frac{t}{2}}(t+1)^{\frac{t+1}{2}}}{t!}$$

The lemma below states our upper bound for $\text{Vol}(\overline{\text{Pos}}_{N,K})$. As we derive our bounds we will find that $\overline{\text{Pos}}_{N,K}$ is always in John's position in a sense we now describe.

Remark 3.1. *We will observe that $\overline{\text{Pos}}_{N,K}$ is a dual body to the convex hull of an isotropic measure on the sphere. Condition of being an isotropic measure with centroid at the origin is actually a “continuous” version of the decomposition of identity in John's Theorem. This point of view is elaborated in [11], and essentially tells us that a section of the cone of nonnegative polynomials $\overline{\text{Pos}}_{N,K}$ is in John's position. This fact will remain valid for cone of nonnegative polynomials supported with arbitrary Newton polytopes.*

Theorem 3.1 above uses a “continuous” version of John's theorem combined with the observation of Ball [2] that conditions of John's theorem are compatible with the Brascamp-Lieb inequality to derive their sharp estimates.

Barvinok and Blekherman [3] used the classical version of John's Theorem to approximate the volume of the convex hull of orbits of compact groups. The classical John's Theorem provides very good approximation for ellipsoid-like bodies but may not be sharp for convex bodies that do not resemble ellipsoids. For instance, as far as we are able to compute Barvinok and Blekherman's Theorem yields an upper bound of order \sqrt{M} for the ratio $\left(\frac{\text{Vol}(\overline{\text{Pos}}_{N,K})}{\text{Vol}(B)}\right)^{\frac{1}{M}}$.

◇

Lemma 3.2.

$$\left(\frac{\text{Vol}(\overline{\text{Pos}}_{N,K})}{|B|}\right)^{\frac{1}{M}} \leq C$$

where M is the dimension of $\overline{\text{Pos}}_{N,K}$, B is the M -dimensional ball with respect to usual inner product, and C is an absolute constant bounded from above by 5.

Proof. We identify (N, K) -homogenous polynomials with the corresponding vector space $P_{N,K}$ of dimension $\mathbb{R}^{\binom{n_1+2k_1-1}{2k_1}\binom{n_2+2k_2-1}{2k_2}}$ where $N = (n_1, n_2)$ and $K = (2k_1, 2k_2)$.

We define a map $\Phi : S^{n_1-1} \times S^{n_2-1} \rightarrow P_{N,K}$ by

$$\Phi(v) = \frac{p_v - r}{\sqrt{\binom{n_1+2k_1-1}{2k_1}\binom{n_2+2k_2-1}{2k_2} - 1}}$$

where p_v is the polynomial corresponding to the vector v as in Lemma 2.13.

It is not hard to prove that Φ is Lipschitz and injective. Now let U be the subspace of $P_{N,K}$ defined by $U = \{p \in P_{N,K} : \langle p, r \rangle = 0\}$. We observe that for all $v \in S^{n_1-1} \times S^{n_2-1}$, $\Phi(v) \in U$ and $\|\Phi(v)\|_2 = 1$.

Now let $\sigma_1 \times \sigma_2$ be the product of uniform measures on S^{n_1-1} and S^{n_2-1} . We define the measure Z on the unit sphere of U , as the push-forward measure of $\sigma_1 \times \sigma_2$ under the map Φ . It follows directly that Z is well-defined, with $\text{Supp}(Z) = \text{Image}(\Phi)$, and satisfies the following property (see, e.g., [16] Theorem 1.19 Chapter 1)

$$\int g dZ = \int g(\Phi) \sigma_1 \times \sigma_2$$

Now for every $q \in U$ we have the following equality

$$\|q\|_2^2 = \int_{S^{n_1-1} \times S^{n_2-1}} q(v)^2 \sigma_1 \times \sigma_2(v) = \int_{S^{n_1-1} \times S^{n_2-1}} M \langle q, \Phi(v) \rangle^2 \sigma_1 \times \sigma_2(v) = \int_{S^{M-1}} M \langle q, v \rangle^2 dZ(v)$$

where $M = \binom{n_1+2k_1-1}{2k_1} \binom{n_2+2k_2-1}{2k_2} - 1$. This simply implies MdZ is an isotropic measure on S^{M-1} !

To compute the centroid of Z let $q = \int_{S^{n_1-1} \times S^{n_2-1}} p_v \sigma_1 \times \sigma_2(v)$. We observe q is invariant under the action of $SO(n_1) \times SO(n_2)$ as defined in Remark 2.1. This observation immediately yields $q = r$. Thus the centroid of Z is the origin. Now using Theorem 3.1 we deduce

$$\text{Vol}(\text{Conv}(\text{Im}(\Phi))^\circ) \leq \frac{M^{\frac{M}{2}} (M+1)^{\frac{M+1}{2}}}{M!}$$

We define $A = \text{Conv}(\{p_v - r : v \in S^{n_1-1} \times S^{n_2-1}\})$ where p_v is the polynomial corresponding to the vector v as defined in Lemma 2.13. We consider A in \mathbb{R}^M , and note that $A = \sqrt{M} \text{Conv}(\text{Image}(\Phi))$. Using the above estimate we have

$$(1) \quad |A^\circ| \leq \frac{M^{\frac{M}{2}} (M+1)^{\frac{M+1}{2}}}{M! M^{\frac{M}{2}}}$$

Now observe that for all $q \in P_{N,K}$ that satisfies $\int_{S^{n_1-1} \times S^{n_2-1}} q = \langle q, r \rangle = 1$ we have

$$q(v) \geq 0 \text{ for all } v \in S^{n_1-1} \times S^{n_2-1} \Leftrightarrow (q - r)(v) \geq -1 \Leftrightarrow \langle r - q, p_v - r \rangle \leq 1$$

Thus $\overline{\text{Pos}}_{N,K} - r = A^\circ$. Hence by 1

$$\left(\frac{|\overline{\text{Pos}}_{N,K}|}{|B|}\right)^{\frac{1}{M}} \leq \left(\frac{M^{\frac{M}{2}}(M+1)^{\frac{M+1}{2}}}{M!M^{\frac{M}{2}}|B|}\right)^{\frac{1}{M}}$$

where B denotes the M dimensional ball.

$$\begin{aligned} \left(\frac{|\overline{\text{Pos}}_{N,K}|}{|B|}\right)^{\frac{1}{M}} &\leq \frac{|B|^{\frac{1}{M}} M^{\frac{1}{2}}}{\frac{M}{e}} \\ \left(\frac{|\overline{\text{Pos}}_{N,K}|}{|B|}\right)^{\frac{1}{M}} &\leq \frac{e}{\sqrt{M}|B|^{\frac{1}{M}}} \leq 5 \end{aligned}$$

■

Remark 3.2. Blekherman derived an upper bound for $\left(\frac{\text{Vol}(\overline{\text{Pos}}_{N,K})}{\text{Vol}(B)}\right)^{\frac{1}{M}}$ in [6] for the usual homogenous polynomial setting with degree fixed. Blekherman's bounds seems sharper than ours for fixed degree homogenous polynomials, i.e., the special case where the underlying Newton polytope is a scaled standard simplex. However, Blekherman's methods do not apply to polynomials supported on more general Newton polytopes. \diamond

The following lemma states our lower bounds for $\text{Vol}(\overline{\text{Pos}}_{N,K})$. The construction carried out in the previous proof seems to indicate a lower bound via discretization and Vaaler's Inequality [22]. For now we give the following lower bound by using the Gauge function.

Lemma 3.3.

$$\left(\frac{|\overline{\text{Pos}}_{N,K}|}{|B|}\right)^{\frac{1}{M}} \geq \frac{1}{\sqrt{16 \max\{n_1, n_2\}(2k_1 + 1)(2k_2 + 1)}}$$

Proof. To derive a lower bound for $\left(\frac{\text{Vol}(\overline{\text{Pos}}_{N,K})}{\text{Vol}(B)}\right)^{\frac{1}{M}}$ we examine $\overline{\text{Pos}}_{N,K} - r$. For any $q \in P_{N,K}$ such that $\langle q, r \rangle = 0$ we observe

$$q \in \overline{\text{Pos}}_{N,K} - r \Leftrightarrow q(v) \geq -1 \text{ for all } v \in S^{n_1-1} \times S^{n_2-1}$$

That is for $f \in U$ and $G_{\overline{\text{Pos}}_{N,K}-r}(f)$ the Gauge Function of $\overline{\text{Pos}}_{N,K} - r$ we have

$$G_{\overline{\text{Pos}}_{N,K}-r}(f) = |\min_{x \in S} f(x)|$$

We set $\|f\|_\infty = \max_{x \in S^{n_1-1} \times S^{n_2-1}} |f(x)|$ and let $S^{M-1} = \{f \in U : \|f\|_2 = 1\}$ then

$$\left(\frac{|\overline{\text{Pos}}_{N,K}| - r}{|B|}\right)^{\frac{1}{M}} = \left(\int_{S^{M-1}} \min_{x \in S^{n_1-1} \times S^{n_2-1}} |f(x)|^{-M} df\right)^{1/M}$$

and

$$\left(\frac{|\overline{\text{Pos}}_{N,K}| - r}{|B|}\right)^{\frac{1}{M}} \geq \left(\int_{S^{M-1}} \|f\|_\infty^{-d} df\right)^{1/d} \geq \int_{S^{M-1}} \|f\|_\infty^{-1} df \geq \left(\int_{S^{M-1}} \|f\|_\infty df\right)^{-1}$$

where the second line of inequalities is derived by consecutive applications of Jensen's inequality. Therefore to prove a lower bound for the volume of $\text{Vol}(\text{Pos}_{N,K})$, it suffices to prove an upper bound for $\int_{S^{M-1}} \|f\|_\infty df$. To this end we invoke Theorem 3.1 from [3] for the compact group $G = SO(n_1) \times SO(n_2)$ and the vector space $V = (R^{n_1})^{\otimes 2k_1} \times (R^{n_2})^{\otimes 2k_2}$. Barvinok's theorem shows that for any $m > 0$ we have

$$\left(\int_{S^{M_1} \times S^{M_2}} f(x)^{2m} \right)^{\frac{1}{2m}} \leq \|f\|_\infty \leq d_m^{\frac{1}{2m}} \left(\int_{S^{n_1-1} \times S^{n_2-1}} f(x)^{2m} \right)^{\frac{1}{2m}}$$

where $d_m \leq \binom{n_1+2k_1m-1}{2k_1m} \binom{n_2+2k_2m-1}{2k_2m}$. This yields

$$\int_{S^{M-1}} \|f\|_\infty df \leq d_m^{\frac{1}{2m}} \int_{S^{M-1}} \left(\int_{S^{n_1-1} \times S^{n_2-1}} f(v)^{2m} dv \right)^{\frac{1}{2m}} df$$

Using Hölder's inequality and Fubini's theorem we have

$$\int_{S^{M-1}} \|f\|_\infty df \leq d_m^{\frac{1}{2m}} \left(\int_{S^{n_1-1} \times S^{n_2-1}} \int_{S^{M-1}} f(v)^{2m} df dv \right)^{\frac{1}{2m}}$$

The average inside the integral is independent of vector v , thus for a fixed v

$$\int_{S^{M-1}} \|f\|_\infty df \leq d_m^{\frac{1}{2m}} \left(\int_{S^{M-1}} \langle f, p_v \rangle^{2m} \right)^{\frac{1}{2m}}.$$

Note that we know $\|p_v\|_2 = \sqrt{M+1}$. So we obtain

$$\begin{aligned} \int_{S^{M-1}} \|f\|_\infty df &\leq d_m^{\frac{1}{2m}} \sqrt{M+1} \left(\frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{2}M)}{\sqrt{\pi}\Gamma(\frac{1}{2}M+m)} \right)^{\frac{1}{2m}} \\ \left(\frac{\Gamma(m+\frac{1}{2})}{\pi} \right)^{\frac{1}{2m}} &\leq \sqrt{m} \text{ and } \left(\frac{\Gamma(\frac{1}{2}M)}{\Gamma(\frac{1}{2}M+m)} \right)^{\frac{1}{2m}} \leq \sqrt{\frac{2}{M}} \\ \int_{S^{M-1}} \|f\|_\infty df &\leq \binom{n_1+2k_1m-1}{2k_1m}^{\frac{1}{2m}} \binom{n_2+2k_2m-1}{2k_2m}^{\frac{1}{2m}} \sqrt{M+1} \sqrt{m} \sqrt{\frac{2}{M}} \end{aligned}$$

We set $h = \max\{n_1, n_2\}$, $m = h(2k_1+1)(2k_2+1)$, for the case $t = (2k_1+1)(2k_2+1) > h$ we have

$$\binom{n_1+2k_1m-1}{2k_1m}^{\frac{1}{2m}} \binom{n_2+2k_2m-1}{2k_2m}^{\frac{1}{2m}} \leq (2k_1m+1)^{\frac{n_1}{2m}} (2k_2m+1)^{\frac{n_2}{2m}} \leq t^{\frac{1}{t}} (th)^{\frac{2}{t}} \leq 4$$

For the case $t = (2k_1+1)(2k_2+1) \leq h$ we write $\binom{n_i+2k_im-1}{2k_im}^{\frac{1}{2m}} \leq (n_i+1)^{\frac{2k_i}{2m}}$ then the rest of the proof follows similarly. Hence we have proved

$$\int_{S^{M-1}} \|f\|_\infty df \leq 4\sqrt{h(2k_1+1)(2k_2+1)}$$

■

Remark 3.3. If we would like the bounds in Lemma 3.2 and Lemma 3.3 to be in terms of the body A that was introduced in the proof of Lemma 3.2 we have

$$c_0 \leq \left(\frac{|A|}{|B|} \right)^{\frac{1}{M}} \leq 4\sqrt{\max\{n_1, n_2\}(2k_1+1)(2k_2+1)}$$

where c_0 is a constant. \diamond

4. THE CONE OF SUMS OF SQUARES

In this section we prove our bounds for $\text{Vol}(\overline{\text{Sq}}_{N,K})$. We start with the upper bound.

Lemma 4.1.

$$\left(\frac{\text{Vol}(\overline{\text{Sq}}_{N,K})}{\text{Vol}(B)} \right)^{\frac{1}{M}} \leq \frac{9}{2} (2^{10}e)^{\frac{k_1+k_2}{2}} \left(\frac{k_1}{n_1+k_1} \right)^{\frac{k_1}{2}} \left(\frac{k_2}{n_2+k_2} \right)^{\frac{k_2}{2}}$$

where c is a constant with $c \leq 5$.

Proof. We define $C = \{p \in U_{N,K} : p+r \in \overline{\text{Sq}}_{N,K}\}$. Let $h_C(f) = \max_{g \in C} \langle f, g \rangle$. We use Urysohn's Lemma [23] in order to bound volume of C . The mean width of C can be written as

$$W_C = 2 \int_{S^{M-1}} h_C(f) d\sigma$$

where $S^{M-1} = \{f \in U_{N,K} : \|f\| = 1\}$. By Urysohn's Lemma we have

$$\left(\frac{\text{Vol}(\overline{\text{Sq}}_{N,K})}{\text{Vol}(B)} \right)^{\frac{1}{M}} = \left(\frac{\text{Vol}(C)}{\text{Vol}(B)} \right)^{\frac{1}{M}} \leq \frac{W_C}{2}$$

Observe that the extreme points of C are of the form $g^2 - r$ where $g \in P_{N,K/2}$ and $\|g\| = 1$. Also observe for $f \in S^{M-1}$, we have $\langle f, r \rangle = \int_S f d\sigma = 0$ that is $\langle f, g^2 - r \rangle = \langle f, g^2 \rangle$. Hence we could write $h_C(f) \leq \max_{g \in P_{N,K/2}, \|g\|=1} \langle f, g^2 \rangle$.

$$\frac{W_C}{2} = \int_{S^{M-1}} h_C(f) d\sigma = \int_{S^{M-1}} \max_{g \in P_{N,K/2}, \|g\|=1} \langle f, g^2 \rangle d\sigma \leq \int_{S^{M-1}} \max_{g \in P_{N,K/2}, \|g\|=1} |\langle f, g^2 \rangle| d\sigma$$

For a fixed f , $\langle f, g^2 \rangle$ is a quadratic form. So Theorem 3.1 of [3] or Barvinok's earlier inequality [4] for $q = \binom{n_1+k_1-1}{k_1} \binom{n_2+k_2-1}{k_2}$ yields

$$\frac{W_C}{2} \leq \int_{S^{M-1}} \left(\int_{S^{D-1}} \langle f, g^2 \rangle^{2q} d\sigma(g) \right)^{\frac{1}{2q}} d\sigma(f) \leq \binom{3q-1}{2q}^{\frac{1}{2q}} \left(\int_{S^{D-1}} \int_{S^{M-1}} \langle f, g^2 \rangle^{2q} d\sigma(f) d\sigma(g) \right)^{\frac{1}{2q}}$$

where $S^{D-1} = \{g \in P_{N,K/2} : \|g\| = 1\}$. Thanks to Reverse Hölder inequalities of J. Duoandikoetxea [9] we know $\|g^2\| \leq 2^{4(k_1+k_2)}$.

We follow the proof of Lemma 3.3 verbatim to arrive at the following estimate

$$\frac{W_C}{2} \leq \left(\frac{3q-1}{2q} \right)^{\frac{1}{2q}} 2^{4(k_1+k_2)} \sqrt{\frac{\binom{n_1+k_1-1}{k_1} \binom{n_2+k_2-1}{k_2}}{\binom{n_1+2k_1-1}{2k_1} \binom{n_2+2k_2-1}{2k_2}}}$$

After this point we apply classical bounds for binomial coefficients, hence

$$\left(\frac{\text{Vol}(\overline{\text{Sq}}_{N,K})}{\text{Vol}(B)} \right)^{\frac{1}{M}} \leq \frac{9}{2} (2^8)^{\frac{k_1+k_2}{2}} \left(\frac{4ek_1}{n_1+k_1} \right)^{\frac{k_1}{2}} \left(\frac{4ek_2}{n_2+k_2} \right)^{\frac{k_2}{2}}$$

■

To prove our lower bound we need the following lemma which was essentially proved by Blekherman as Lemma 5.3 at [6]

Lemma 4.2. (*Blekherman*)

$$\text{Sq}_{N,K}^{d*} \subseteq \text{Sq}_{N,K}$$

where $\text{Sq}_{N,K}^{d*}$ is the dual cone with respect to the differential metric.

Lemma 4.3.

$$\left(\frac{\text{Vol}(\overline{\text{Sq}}_{N,K})}{\text{Vol}(B)} \right)^{\frac{1}{M}} \geq c \prod_{i=1}^m \left(\frac{1}{2k_i + \frac{n_i}{2}} \right)^{\frac{k_i}{2}}$$

Proof.

$$\langle T(r), r \rangle_D = \langle r, r \rangle_D = C_{N,K} \langle r, r \rangle = C_{N,K}$$

$$\overline{\text{Sq}}_{N,K} = \{p \in \text{Sq}_{N,K} : \langle p, r \rangle = 1\} := \{p \in \text{Sq}_{N,K} : \langle p, r \rangle_D = C_{N,K}\}$$

$$A = \overline{\text{Sq}}_{N,K} - r := \{p \in P_{N,K} : p + r \in \text{Sq}_{N,K} \text{ and } \langle p, r \rangle_D = 0\}$$

$$A_d^\circ = \{q \in P_{N,K} : \langle q, r \rangle_D = 0, \langle q, p \rangle_D \leq 1 \ \forall p \in A\}$$

$$C_{N,K}^{-1}r - A_d^\circ = \{q \in P_{N,K} : \langle q, r \rangle_D = 1, \langle q, p \rangle_D \geq -1 \ \forall p \in A\}$$

$$C_{N,K}^{-1}r - A_d^\circ = \{q \in P_{N,K} : \langle q, r \rangle_D = 0, \langle q, p \rangle_D \leq 0 \ \forall p \in \overline{\text{Sq}}_{N,K}\}$$

Observe that for any $f \in P_{N,K}$

$$\langle f, g \rangle_D \geq 0 \ \forall g \in \overline{\text{Sq}}_{N,K} \Leftrightarrow \langle f, g \rangle_D \geq 0 \ \forall g \in \text{Sq}_{N,K}$$

thus $C_{N,K}^{-1}r - A_d^\circ = B \subseteq \text{Sq}_{N,K}^{d*} \subseteq \text{Sq}_{N,K}$. Hence $B \subseteq \text{Sq}_{N,K} = C_{N,K}^{-1}\overline{\text{Sq}}_{N,K}$. By the Reverse Santalo inequality [10] we have

$$c \leq \left(\frac{\text{Vol}(A)}{\text{Vol}(B_D)} \right)^{\frac{1}{M}} \left(\frac{\text{Vol}(A_d^\circ)}{\text{Vol}(B_D)} \right)^{\frac{1}{M}} \leq \left(\frac{\text{Vol}(\overline{\text{Sq}}_{N,K})}{\text{Vol}(B_D)} \right)^{\frac{1}{M}} \left(\frac{\text{Vol}(C_{N,K}^{-1}\overline{\text{Sq}}_{N,K})}{\text{Vol}(B_D)} \right)^{\frac{1}{M}}$$

$$\sqrt{c \cdot C_{N,K}} \leq \left(\frac{\text{Vol}(\overline{\text{Sq}}_{N,K})}{\text{Vol}(B_D)} \right)^{\frac{1}{M}}$$

Using the bounds in Lemma 2.11 completes the proof. \blacksquare

5. THE CONE OF POWERS OF LINEAR FORMS

This section develops quantitative bounds on the cone of even powers of linear forms. More precisely let $\bar{L}_{N,K} := \{p \in L_{Q_{N,K}} : \langle p, r \rangle = 1\}$, we prove upper and lower bounds on the volume of $\bar{L}_{N,K}$.

Now let us consider the image of p_v (as in Lemma 2.13) under the map T :

- (1) $\langle T(p_v), r \rangle_D = C_{N,K}$
- (2) For every $f \in P_{N,K}$ we have

$$\langle f, T(p_v) \rangle_D = C_{N,K} \langle f, p_v \rangle = C_{N,K} f(v)$$

Since for all $f \in P_{N,K}$, $\langle f, C_{N,K} K_v \rangle_D = C_{N,K} f(v)$ we have $T(p_v) = C_{N,K} K_v$. Now let $\acute{A} = \{p_v : v \in S\}$ and $A = \text{Conv}(\acute{A})$. If we define the map Φ as in the non-negative polynomials section we observe $A = \sqrt{\dim(P_{N,K})} \text{Conv}(\text{Image}(\Phi))$. By Krein-Milman theorem and linearity of T we have

$$\bar{L}_{N,K} = \text{Conv}(C_{N,K} K_v : v \in S) = \text{Conv}(T(\acute{A})) = T(\text{Conv}(\acute{A})) = T(A)$$

This implies that $\text{Vol}(\bar{L}_{N,K}) = |\det(T)| \text{Vol}(A)$. Therefore, from Remark 3.3 and the bounds derived in Lemma 2.11, we deduce the following estimate on the volume of $\bar{L}_{N,K}$.

Lemma 5.1.

$$c_0 \prod_{i=1}^m \left(\frac{\frac{1}{k_i}}{2 + \frac{n_i}{2k_i}} \right)^{\frac{k_i}{2}} \leq \left(\frac{|\bar{L}_{N,K}|}{|B|} \right)^{\frac{1}{M}} \leq 4 \sqrt{\max\{n_1, n_2\} (2k_1 + 1)(2k_2 + 1)} \prod_{i=1}^m \left(\frac{1}{1 + \frac{n_i}{2k_i}} \right)^{\frac{k_i}{2}}$$

where c_0 is a positive constant and $c_0 \leq 5$.

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